Mid-Semester Examination
II Semester, 2011-2012

M. Math I Year
Complex Analysis

The questions carry a total of 110 marks. The maximum you can score is 100.
Time limit: 3 hrs.

1. Find a one-to-one holomorphic map $F : U \to U$ which is onto U and satisfies the condition
$f(1/2) = 1/2$. [10]

2. If Ω is a region and $f \in H(\Omega)$ with $\mathrm{Re}(f) = [\mathrm{Im}(f)]^2$ on Ω show that f is a constant on Ω. [10]

3. For the following functions find the nature of singularity at 0. In case 0 is a pole find the residue. If it is an essential singularity find a point in $B(0,1)$ whose image is i.

a) $f(z) = \frac{\tan(z)}{z^4}$ $\quad$ b) $f(z) = \cos(\frac{1}{z})$ $\quad$ c) $f(z) = \frac{1-\cos(z)}{z^2}$ $\quad$ $[10 + 10 + 5]$

4. Evaluate $\int\limits_0^\infty \frac{x^2-1}{x^4-5x^3+7x^2-5x+6} dx$ by the method of residues. [30]
[HINT: the denominator is zero at some small integers].

5. Prove or disprove: given a sequence of complex numbers $\{c_n\}$ there is a function f holomorphic in U such that $f^{(n)}(0) = c_n \forall n \geq 0$. [10]

6. Let $f \in H(\Omega)$ and $g = \frac{f'}{f}$. Let f have a zero at $a \in \Omega$. Prove that the residue of g at a is the order of zero of f at a. [15]

7. Assuming the result of problem 6) above prove the following:
if $f \in H(B(a, 2r))$ then the number of zeroes of f in $\{z : |z - a| \leq r\}$ counted according to multiplicites equals $\int\limits_\gamma \frac{f'}{f}(z) dz$ times $2\pi i$ assuming that f has no zero on γ, where $\gamma(t) = a + re^{2\pi it}(0 \leq t \leq 1)$. [10]